Taylor is prime

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Varieties and Maltsev conditions

Example

The variety $\mathcal{SET}$ is the class of all sets with no basic operations.

Example

The variety $\mathcal{SG}$ is the class of all semigroups $(S; \circ)$ defined by associativity.

Example

The variety $\mathcal{SLAT}$ is the class of all semilattices $(S; \wedge)$ defined by associativity, commutativity and idempotency.

$\mathcal{SET}$ and $\mathcal{SG}$ are not the same, but every semigroup can be turned into a set, and every set can be turned into a semigroup by defining $x \circ y = x$.

Semigroups cannot be turned into semilattices, because semilattices satisfy the identity $x \wedge y = y \wedge x$ and such operation cannot be defined as a semigroup term.

Interpretation of varieties

Definition (W.D. Neumann; 1974)

Let Γ be a set of identities over a signature. We say that Γ **interprets in a variety** $\mathcal{K}$ if by replacing the operation symbols in Γ by some term expressions of $\mathcal{K}$, the so obtained set of identities holds in $\mathcal{K}$.

Definition

A **variety** $\mathcal{K}_1$ **interprets in a variety** $\mathcal{K}_2$, denoted as $\mathcal{K}_1 \leq \mathcal{K}_2$, if there is a set of identities Γ that defines $\mathcal{K}_1$ and interprets in $\mathcal{K}_2$.

- The varieties of Boolean algebras and rings are equi-interpretable.
- The varieties of Sets and Semigroups are equi-interpretable.
- The variety of sets interprets in any other variety.
- Every variety interprets in the variety of trivial algebras ($x \approx y$).
- Constants c are modelled by unary operations satisfying $c(x) \approx c(y)$.
- The iterpretability relation $\leq$ is a quasi-order on the class of varieties.

Lattice of interpretability types

Theorem

The class of varieties modulo equi-interpretability forms a bounded lattice, **the lattice of interpretability types**, *with* $\overline{\mathcal{V}} \vee \overline{\mathcal{W}} = \overline{\mathcal{V} \amalg \mathcal{W}}$ *and* $\overline{\mathcal{V}} \wedge \overline{\mathcal{W}} = \overline{\mathcal{V} \otimes \mathcal{W}}$.

Definition

The **coproduct** of the varieties $\mathcal{V} = \operatorname{Mod}\Sigma$ and $\mathcal{W} = \operatorname{Mod}\Delta$ in disjoint signatures is the variety $\mathcal{V} \amalg \mathcal{W} = \operatorname{Mod}(\Sigma \cup \Delta)$.

Definition

The **varietal product** of $\mathcal{V}$ and $\mathcal{W}$ is the variety $\mathcal{V} \otimes \mathcal{W}$ of algebras $\mathbf{A} \otimes \mathbf{B}$ for $\mathbf{A} \in \mathcal{V}$ and $\mathbf{B} \in \mathcal{W}$ whose

- universe is $A \times B$,
- basic operations are $s \otimes t$ acting coordinate-wise for each pair of n-ary terms of $\mathcal{V}$ and $\mathcal{W}$.

O. Garcia, W. Taylor (1984): Lattice of interpretability types of varieties

- minimal element: sets (equi-interpretable with semigroups)
- maximal element: trivial algebras
- the class of idempotent varieties form a sublattice
- the class of finitely presented varieties forms a sublattice
- the class of varieties defined by linear equations forms a join sub-semilattice
- not modular
- meet prime elements: boolean algebras, lattices, semilattices
- meet irreducible elements: groups
- join prime elements: commutative groupoids, trivial algebras

J. Mycielski (1977): Lattice of interpretability types of first order theories

- local interpretability
- distributive

Maltsev conditions

Definition

A **strong Maltsev condition** is a finite set Γ of equations. A variety $\mathcal{V}$ satisfies Γ iff Γ interprets in $\mathcal{V}$. Thus strong Maltsev conditions are principal filters generated by finitely based varieties in the lattice of interpretability types.

Examples: having a majority term, semilattice term, Siggers term, Olšák term, Maltsev term, etc.

Definition

A **Maltsev condition** is a descending chain $\Gamma_0 \geq \Gamma_1 \geq \Gamma_2 \geq \dots$ of strong Maltsev conditions. A variety $\mathcal{V}$ satisfies it if $\Gamma_i \leq \mathcal{V}$ for some $n \in \mathbb{N}$. Thus Maltsev conditions are filters in the lattice of interpretability types.

Examples: having a near-unanimity term, Taylor term, Jónsson terms, Gumm terms, edge term, Hagemann-Mitschke terms, etc.

Conjecture (O. Garcia, W. Taylor; 1984)

2-permutability is prime in the lattice of interpretability types of varieties.

- In 1996 S. Tschantz announced a proof of the conjecture. However, his proof has remained unpublished.
- K. Kearnes and S. Tschantz: 2-permutability is prime in the lattice of interpretability types of **idempotent** varieties. There are no similar results known for n-permutability when $n > 2$.
- J. Opršal: for any $n \geqslant 2$, n-permutability is prime in the lattice of interpretability types of **linear** varieties.

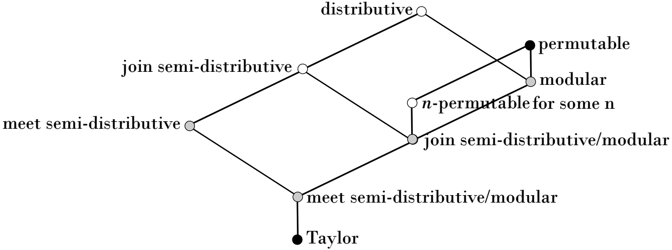
Theorem (G. Gyenizse, M. M. and L. Zádori; 2020)

For $n \geqslant 5$, n-permutability is **not prime** *in the lattice of interpretability types of varieties (polymorphisms of the hexagon poset and majority).*

Theorem (G. Gyenizse, M. M. and L. Zádori; 2022)

2-permutability **is prime** *in the lattice of interpretability types of varieties.*

Maltsev filters of varieties



- prime Maltsev filters:
  - congruence 2-permutable, $m(x, y, y) \approx m(y, y, x) \approx x$
  - Taylor term, non-trivial idempotent Maltsev condition
- non-prime Maltsev filters:
  - congruence n-permutable for some n, Hagemann-Mitschke terms
  - (congruence) distributive $=$ join semi-distributive and modular
  - congruence join semi-distributive (K. Kearnes and E. W. Kiss)

Main Theorem

Theorem (W. Taylor, 1977; J. Olšák, 2017)

For any variety $\mathcal{V}$ the following are equivalent

- $\mathcal{V}_{\mathsf{id}} \not\leq \mathcal{SET}$,
- *satisfies a non-trivial idempotent Maltsev condition,*
- *has a Taylor-term:* $t(x,\dots,x) \approx x$ *and* $t(\dots,x,\dots) \approx t(\dots,y,\dots)$,
- *has an Olšák term* $t(x,x,x,x,x,x) \approx x$ *and* $t(x,y,y,y,x,x) \approx t(y,x,y,x,y,x) \approx t(y,y,x,x,x,y)$.

Theorem

The filter of Taylor varieties is prime in the lattice of interpretability types.

Approach: Given two non-Taylor varieties $\mathcal{V}$ and $\mathcal{W}$, find a compatible digraph $\mathbb{G}$ in both $\mathcal{V}$ and $\mathcal{W}$ that does not admit a Taylor polymorphism.

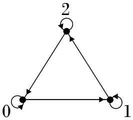
The reflexive 3-cycle $\mathbb{C}$

Theorem

A variety is Taylor iff all its reflexive anti-symmetric digraphs are cycle free.

Definition

Let $\mathbb{C} = (\{0, 1, 2\}; \to)$ be the reflexive directed 3-cycle.



Lemma

There are six essential polymorphisms of $\mathbb{C}$, the constants and the automorphisms. Thus $\mathbf{C} = (C; \mathrm{Pol}(\mathbb{C}))$ generates a non-Taylor variety.

Question: Which non-Taylor variety (reflexive anti-symmetric digraph with a cycle) is the strongest (satisfies the most Maltsev conditions)?

Non-Taylor varieties with non-trivial Maltsev conditions

Proposition

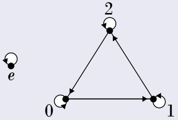
- $\mathcal{SET} \lnsim \mathrm{Pol}(\mathbb{C})$ *because of the Maltsev condition* $u(x) \approx u(y)$
- $\mathrm{Pol}(\mathbb{C}) \lnsim \mathrm{Pol}(\mathbb{C}^2)$ *because of the Maltsev condition*

$$f(f(x,y), f(y,z)) \approx y$$

satisfied by the polymorphism $f(\overline{x_1x_2}, \overline{y_1y_2}) = \overline{x_2y_1}$ *of* $\mathbb{C}^2$.

- $\mathrm{Pol}(\mathbb{C}+1) \not\lesssim \mathrm{Pol}(\mathbb{C}^K)$ *because of the Maltsev condition*

$$e(x) \approx e(y), \quad t(e(x), y) \approx t(y, e(x)) \approx y.$$



- $\mathrm{Pol}(\mathbb{G}) \leq \mathrm{Pol}(4\mathbb{C}+4)$, *for the reflexive 4-cycle digraph* $\mathbb{G}$

Lemma

A variety is non-Taylor iff it has a compatible reflexive digraph $\mathbb{F}$ that has $\mathbb{C}$ as a retract.

- Let $\mathcal{V}$ be a non-Taylor variety
- $\mathbf{F}$ the free algebra in $\mathcal{V}$ freely generated by $\{x, y, z\}$
- ϱ the subalgebra of $\mathbf{F}^2$ generated by $\{xx, yy, zz, xy, yz, zx\}$
- $\mathbb{F} = (F; \varrho)$ is reflexive digraph
- $u \to v$ in $\mathbb{F}$ iff there exists a 6-ary term t in $\mathcal{V}$ so that

$$u(x, y, z) \approx t(x, y, z, x, y, z), \quad t(x, y, z, y, z, x) \approx v(x, y, z)$$

- in particular, if $u \to v$, then $u(x, x, x) \approx v(x, x, x)$
- $\mathbb{F}$ has as many (strong) components as there are unary terms in $\mathcal{V}$
- $\mathcal{V}_{\text{id}} \preceq \mathcal{SET}$, so there is a graph homomorphism form the idempotent component of $\mathbb{F}$ to $\mathbb{C}$
- $\mathbb{C}$ is reflexive, so this can be extended to an $\mathbb{F} \to \mathbb{C}$ homomorphism
- so $\mathbb{C}$ is a graph retract of $\mathbb{F}$
- $\mathcal{V} \preceq \text{Pol}(\mathbb{F})$ and $\text{Pol}_{\text{id}}(\mathbb{F}) \preceq \mathcal{SET}$

Theorem

For any variety $\mathcal{V}$ the following are equivalent:

- *$\mathcal{V}$ is non-Taylor,*
- *there are sets K_t $(t \in T)$, not all empty, such that $\dot{\bigcup}_{t \in T} \mathbb{C}^{K_t}$ is a compatible digraph in $\mathcal{V}$,*
- *for any sufficiently large infinite cardinals κ and τ the digraph $\dot{\bigcup}_{\mu \leqslant \kappa} \tau \mathbb{C}^{\mu}$ is a compatible digraph in $\mathcal{V}$.*

Corollary

The filter of Taylor varieties is prime in the lattice of interpretability types.

Problem

Describe the interpretability lattice for the varieties generated by the disjoint union of $\mathbb{C}$-powers.

Problem

Is there a direct connection between $\mathbb{C}$ and the Olšák term?